2015-2016 Mathematics Competition Practice Session #2

Hagerstown Community College: STEM Club

October 9, 2015 12:00 pm – 1:30 pm Student Center, STC-167

1. Warm-up (2014 AMC 8 #5):

Margie's car can go 32 miles on a gallon of gas, and gas currently costs $\$4$ per gallon. How many miles can Margie drive on $\$20$ worth of gas?

2. Beginner (2014 AMC 12B #17):

Let P be the parabola with equation $y = x^2$ and let $Q = (20, 14)$. There are real numbers r and s such that the line through Q with slope m does not intersect P if and only if $r < m < s$.

What is $r + s$?

$\textbf{(A)}\ 1 \qquad \textbf{(B)}\ 26 \qquad \textbf{(C)}\ 40 \qquad \textbf{(D)}\ 52 \qquad \textbf{(E)}\ 80$

3. Intermediate (2013 AIME II #2):

Positive integers a and b satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of $a + b$.

4. **Advanced (2014 Harvard-MIT Mathematics Tournament #6):**

Given w and z are complex numbers such that $|w+z|=1$ and $|w^2+z^2|=14$, find the smallest possible value of $|w^3+z^3|$. Here $|\cdot|$ denotes the absolute value of a complex number, given by $|a+bi|=\sqrt{a^2+b^2}$ whenever a and b are real numbers.

5. **200-Level ("Gabriel's Horn"):**

Find a real-valued function such that, when one rotates the graph of said function in the xy-plane about the x-axis, the volume is finite but the surface area is infinite. Prove that the function has these two properties.

Solutions

1. Let m be the number of miles travelled and g be the worth of the particular quantity of gas used, then we have that $m = 32$ when $g = \$4$, thus $m = 8g$, so if $g = \$4(5) = \$20$, then $m = 32(5) = \mathbf{160\ miles.}$

   A more realistic methodology would be to say Margie has enough money for $\frac{20}{4} = 5$ gallons of gas and therefore can travel $32(5) = \mathbf{160}$ **miles**.

2. By the equation of a line $y - y_1 = m(x - x_1)$ we have that $y = m(x - 20) + 14$ is the equation of the line through Q with slope m. P not intersecting with the line through Q means that, when the equation of the line through Q and the equation of the parabola P are set equal to each other, there are no real solutions. In other words,

   $$x^2 \neq m(x - 20) + 14 \quad , \quad x \in \mathbf{R}$$

   What this then comes down to is looking at what m makes that true. If the two are not equal for real x then we have that

   $$x^2 - mx + 20m - 14 \neq 0 \quad , \quad x \in \mathbf{R}$$

   Recall the quadratic formula:

   $$ax^2 + bx + c = 0 \quad \leftrightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   One should also recall that the “discriminant”, *i.e.* the portion under the square root, must be positive in order to have real solutions. Thus, if we find all m such that the discriminant is negative, we have found the slopes where the line and the parabola fail to intersect.

   Upon identifying a to be 1, b to be $-m$, and c to be the entire constant term, $20m - 14$, we let the discriminant be less than zero:

   $$(-m)^2 - 4(1)(20m - 14) = m^2 - 80m + 56 < 0$$

   Now, we use one of *Vieta’s formulas* (see the References section at the end of this document) one the polynomial in m:

   $$r + s = -\left(\frac{b}{a}\right) = -\left(-\frac{80}{1}\right) = \mathbf{80}$$

   Note that we could avoid invoking Vieta’s formulas here if we were only to use a calculus-based solution as follows …

   The slope of the parabola is $2x$ (taking the derivative yields this). Solving for our slope m we find that the slope is also equal to

   $$\frac{14 - x^2}{20 - x}$$

Setting these two values for the slope equal to one another yields

$$x^2 - 40x + 14 = 0 \quad \to \quad x = 20 \pm \sqrt{386}$$

Summing these two possible values yields 40. However, $m = 2x$, so our solution is **80**.

3. Recall the property of logarithms that $b^{\log_b(x)} = x$ for any base b, thus

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0 \quad \leftrightarrow \quad \log_{2^a}(\log_{2^b}(2^{1000})) = 1 \quad \leftrightarrow \quad \log_{2^b}(2^{1000}) = 2^{\mathrm{a}} \quad \leftrightarrow \quad 2^{1000} = (2^b)^{2^a}$$

Now, using the property that $\left(a^b\right)^c = a^{bc}$, we have that

$$2^{1000} = 2^{b2^a} \quad \leftrightarrow \quad 1000 = b2^a$$

We need to find all a, b satisfying this, thus we can find the largest a (or b) and work backwards. Then, what is the largest value of a such that 1000 is divisible by 2^a? One can simply check them all and find the answer is $a = 3$. For $a = 3$ we have that $b = 125$. Similarly, if $a = 2$ then $b = 250$ and if $a = 1$ then $b = 500$. The sum of all of these is our answer:

$$3 + 125 + 2 + 250 + 1 + 500 = \mathbf{881}$$

4. Let $w + z = \alpha$ and $wz = \beta$. Then $w^2 + z^2 = \alpha^2 - 2\beta$ (square both equations and combine) and similarly $w^3 + z^3 = \alpha^3 - 3\alpha\beta$. We thus have that $|\alpha| = 1$ and $|\alpha^2 - 2\beta| = 14$, and we must minimize $|\alpha^3 - 3\alpha\beta|$. We begin by factoring out a term of α, as $|\alpha| = 1$, thus we may minimize $|\alpha^2 - 3\beta|$ and this will suffice.

Knowing that, without loss of generality due to the fact that we may always appropriately rotate our complex plane and our equations lie on circles, $\alpha = 1$ and $\alpha^2 - 2\beta = 14$, we may solve for $\alpha^3 - 3\beta$ by noting that

$$1 - 2\beta = 14 \quad \to \quad \beta = -\tfrac{13}{2}$$

Therefore,

$$\alpha^3 - 3\beta = 1 - \left(-\frac{13}{2}\right) = \mathbf{\frac{41}{2}}$$

One who has taken complex analysis will probably recognize that the possible values of our squared equation lie on a circle of radius 14 centered at the origin and that the possible values of our cubed equation are the image of that circle under a dilation of $3/2$. This might help to fully understand the note about not losing generality.

In practice, one would probably be a bit looser in their thinking of the details, however.

5. While one *could* derive this by writing out general integrals for surface area and volume, then going through a great deal of not only tedious work but careful consideration (I have done this before), the real intention of this question was to create some discussion. Can such a function exist? If so, what is that function?

The answer is something called Gabriel's Horn or Torricelli's trumpet, named after the Archangel Gabriel and his horn, or the Italian mathematician who first discovered it in the 1800s.

The function is simple:

$$f(x) = \frac{1}{x} \quad , \quad x \in [1, \infty)$$

The restricted domain is solely to avoid the asymptote. As long as we avoid that asymptote and do not have our bounds at both positive and negative infinity, we still have the same basic result, but the volume is, of course, a different numerical value.

To prove our assertions, let us integrate:

$$V = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx = \pi \lim_{n\to\infty} \int_1^{n} \left(\frac{1}{x}\right)^2 dx = \pi \lim_{n\to\infty} \left[\frac{x^{-1}}{-1}\right]_1^n = \pi \lim_{n\to\infty} \left(-\frac{1}{n} - (-1)\right) = \pi(1) = \pi$$

&

$$A = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1 + \left(f'(x)\right)^2} dx = 2\pi \lim_{n\to\infty} \int_1^{n} \frac{1}{x}\sqrt{1 + (-x^{-2})^2}\, dx$$

Inspecting the integral a bit, ignoring the limit for now, shows that the integral is strictly greater than the same integral without the root part. This is done by the "comparison test for improper integrals", because the integrand $\frac{1}{x}\sqrt{1 + (-x^{-2})^2} > 1 > \frac{1}{x}$ on the interval [0,1), we have that if the smaller diverges, then so does the larger (*i.e.* if ignoring the radical yields a diverging integral, then our integral diverges).

So, we look at the compared integral.

$$2\pi \lim_{n\to\infty} \int_1^{n} \frac{1}{x}\, dx = 2\pi \lim_{n\to\infty} \ln(n) = 2\pi(\infty) = \infty$$

Since this smaller integral is infinite, so too is the area integral. Thus, the surface area is indeed infinite.

There is a great deal of interesting history behind this including an apparent paradox. Moreover, it is provable that there can exist no situation where an object has an infinite volume but a finite surface area. Perhaps most interestingly, though, is that if we translate this to real world terms (not that this is in any way possible in the real world due to physical limitations not taken into the mathematics), we get the following:

Painter's Paradox: Seemingly, one could fill the inside of a horn of this shape with paint but would require an infinite amount of paint and time to paint the outside. (This assumes that the paint does not diminish in thickness rapidly tending toward a zero thickness.)

The other "paradox" mentioned actually caused dispute among some thinkers of the past, most notably Galileo.

The Illegitimate Paradox of the XY Plane: Taking an infinite section of the xy plane and rotating it about the x-axis should not be able to yield a finite volume, or so it seems. The incorrectness of the notion that this is impossible relates nicely to the divergence of the harmonic series, as summing the radii of the disks which are “stacked” to form the horn requires one run into the harmonic series, but in actuality you should sum the areas, which involve summing the series $1/x^2$, which converges (as does any power greater than 1).

Disproving the converse is somewhat more complicated and general than proving the original case. However, it is possible for a bright undergraduate with some comfortability with the limit superior (supremum) to do. In fact, ...

Exercise: *Prove that there exists no continuous function* $f: [1, \infty) \to [0, \infty)$ *such that the solid of revolution of the graph* $y = f(x)$ *about the* x*-axis has the property that it has finite surface area but infinite volume. (In other words, prove that if the surface area is finite, then the volume is finite.)*

(One can check their proof against the one on Wikipedia at https://en.wikipedia.org/wiki/Gabriel%27s_Horn#Converse. Of course, your proof may be different and still correct, but it is likely that a reasonable person will develop a proof that is essentially the same as the one linked above.)

Reference: Vieta's Formulas:

Vieta's formulas can be stated as follows:

For the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, all of the following hold:

$$\begin{cases} x_1 + x_2 + \cdots + x_{n-1} + x_n = -\dfrac{a_{n-1}}{a_n} \\ (x_1x_2 + x_1x_3 + \cdots + x_1x_n) + (x_2x_3 + x_2x_4 + \cdots + x_2x_n) + \cdots + x_{n-1}x_n = \dfrac{a_{n-2}}{a_n} \\ \quad\vdots \\ x_1x_2\ldots x_n = (-1)^n\dfrac{a_0}{a_n}. \end{cases}$$

In other terms:

$$\sum_{1\le i1<i2<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k} = (-1)^k\frac{a_{n-k}}{a_n}$$

Quoting the Art of Problem Solving website directly, here is some information, including a derivation. The proof of these formulas are quite beautiful.

Vieta's Formulas were discovered by the French mathematician François Viète.

Vieta's Formulas can be used to relate the sum and product of the roots of a polynomial to its coefficients. The simplest application of this is with quadratics. If we have a quadratic $x^2 + ax + b = 0$ with solutions p and q, then we know that we can factor it as

$$x^2 + ax + b = (x-p)(x-q)$$

(Note that the first term is x^2, not ax^2.) Using the distributive property to expand the right side we get

$$x^2 + ax + b = x^2 - (p+q)x + pq$$

We know that two polynomials are equal if and only if their coefficients are equal, so $x^2 + ax + b = x^2 - (p+q)x + pq$ means that $a = -(p+q)$ and $b = pq$. In other words, the product of the roots is equal to the constant term, and the sum of the roots is the opposite of the coefficient of the x term.

A similar set of relations for cubics can be found by expanding

$$x^3 + ax^2 + bx + c = (x-p)(x-q)(x-r).$$

We can state Vieta's formula's more rigorously and generally. Let $P(x)$ be a polynomial of degree n, so $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where the coefficient of x^i is a_i and $a_n \neq 0$ As a consequence of the Fundamental Theorem of Algebra, we can also write $P(x) = a_n(x-r_1)(x-r_2)\cdots(x-r_n)$, where r_i are the roots of $P(x)$. We thus have that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - r_1)(x - r_2) \cdots (x - r_n).$$

Expanding out the right hand side gives us

$$a_n x^n - a_n (r_1 + r_2 + \cdots + r_n) x^{n-1} + a_n (r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n) x^{n-2} + \cdots + (-1)^n a_n r_1 r_2 \cdots r_n.$$

The coefficient of x^k in this expression will be the kth symmetric sum of the r_i.

We now have two different expressions for $P(x)$. These must be equal. However, the only way for two polynomials to be equal for all values of x is for each of their corresponding coefficients to be equal. So, starting with the coefficient of x^n, we see that

$$\begin{aligned} a_n &= a_n \\ a_{n-1} &= -a_n (r_1 + r_2 + \cdots + r_n) \\ a_{n-2} &= a_n (r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n) \\ &\vdots \\ a_0 &= (-1)^n a_n r_1 r_2 \cdots r_n \end{aligned}$$

More commonly, these are written with the roots on one side and the a_i on the other (this can be arrived at by dividing both sides of all the equations by a_n).

If we denote σ_k as the kth symmetric sum, then we can write those formulas more compactly as

$$\sigma_k = (-1)^k \cdot \frac{a_{n-k}}{a_n}, \text{ for } 1 \le k \le n.$$