# 2015-2016 Mathematics Competition Practice Session 3

Hagerstown Community College: STEM Club

October 23, 2015 12:00 pm - 1:00 pm STC-170

This session we have three problems indicative of the competition to work on, each one progressively becoming more difficult. Before beginning problems a brief presentation will be given regarding the nature of number theory. We also have two additional problems, one of which utilizes calculus and the other of which is oddly similar and equally as intriguing. Neither are expected to be solved, but good discussion can be had, and moreover the solutions can be presented. Finally, we have a challenge problem regarding derivatives. Whoever first submits a correct solution will get some sort of prize.

## Beginner (AMC 12B 2011, No. 15):

How many positive two-digit integers are factors of $2^{24} - 1$?

## Intermediate: (2010 AIME I, No. 12):

Let $m \leq 3$ be an integer and let $S = \{3, 4, 5, \ldots, m\}$. Find the smallest value of $m$ such that for every partition of $S$ into two subsets, at least one of the subsets contains integers $a, b, c$ (not necessarily distinct) such that $ab = c$.

*Note: A partition of $S$ is a pair of sets $A, B$ such that $A \cap B = \varnothing$, $A \cup B = S$.*

## Advanced (IMO 1990):

Determine all positive integers $n$ such that $\frac{2^n+1}{n^2}$ is an integer.

## Challenge (Source To Be Revealed):

Let $k \in \mathbb{Z}^+$ be fixed.

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{1}{x^k - 1}\right) = \frac{P_n(x)}{(x^k - 1)^{n+1}}$$

where $P_n(x)$ is a polynomial. Find $P_n(1)$.

## Discussion Question ("Buffon's Needle Problem"):

What is the probability that a dropped needle will cross one of a set of equally spaces parallel lines? In other words, if we had a floor composed of rectangular boards of equal width, and we dropped a needle, then what is the probability that the needle would fall such that it lay across two boards.

## Discussion Question ("Kakeya's Needle Problem")

What is the minimum area $A$ of a region $D$ in the plane in which a needle of length $\ell = 1$ can be turned by one revolution ($360°$)? (The needle is taken to be a line segment of zero width.)

## Beginner: *Solution*

One should note quite quickly that this expression can be factored via difference of squares.

$$\begin{aligned}
2^{24} - 1 &= (2^{12} + 1)(2^{12} - 1) \\
&= (2^{12} + 1)(2^6 + 1)(2^6 - 1) \\
&= (2^{12} + 1)(2^6 + 1)(2^3 + 1)(2^3 - 1) \\
&= (2^{12} + 1) \cdot 65 \cdot 9 \cdot 7
\end{aligned}$$

Now we will apply the sum of cubes formula to the remaining term:

$$\begin{aligned}
2^{12} + 1 &= (2^4 + 1)(2^8 - 2^4 + 1) \\
&= 17 \cdot 241
\end{aligned}$$

One can check that 241 is indeed a prime number, so we can turn to the smaller factors, namely $3^2, 5, 7, 13, 17$. We want to multiply these factors together to get another two digit factor, so we can do $17 \cdot 3$, $17 \cdot 5$, $13 \cdot 3$, $13 \cdot 5$, $13 \cdot 7$, $7 \cdot 3$, $7 \cdot 5$, $7 \cdot 3^2$, $5 \cdot 3$, and $5 \cdot 3^2$. Any other multiplication of factors will either be redundant or yield a number that is not 2 digits. So, we count the number of factors in total. We had found 5 factors initially, only 2 of which were 2 digits. Then, we found 10 more 2 digit factors by the multiplication method described before. Hence, we have

$$2 + 10 = \boxed{12}$$

## Intermediate: ***Solution***

It is hard to know where to start with problems like these. Perhaps we can just try to guess an $m$. So, let us try a value; $m = 10$ seems nice. In that case we have many partitions of $S$. One example is to partition $S$ into $S \cap \{1,2,3,4,5\}$ and $S \cap \{6,7,8,9,10\}$. Here we clearly have that the desired condition is not satisfied. One has to make the observation that in order for the desired property to hold we must have that $m$ has many factors.

It is difficult to verbally explain how one makes the jump, but one can reasonably conclude that $m = 243$ might work, because it has nice factors, namely $3 \times 81$ and $9 \times 27$ work.

Now, let us partition 243 and try to find a situation in which the partition does *not* give us the desired property. We will partition $S$ into $A$ and $B$, and without loss of generality, we will place $3 \in A$, then we must put $9 \in B$, and so we need $81 \in A$, and we therefore select $27 \in B$. This guarentees that $ab \neq c$, but then we have 243 left, and we cannot place that into *either* set, so we know that $m \leq 243$.

For $m < 243$ we have that $S$ is partitioned into $S \cap \{3,4,\ldots 8, 81, 82, \ldots 242\}$ and $S \cap \{9, 10, \ldots 80\}$ and in neither set are there values $ab = c$, since the maximum value of the lower section of the first partition is 8, which is strictly less than any value, $3^2$ to $8^2$, which is similarly less than 81, and similarly $9^2$ to $80^2$ is strictly less than 80. Hence, the minimum $m$ must be 243, as proposed.

$$\boxed{243}$$

## Advanced: *Solution*

After checking small values of $n$ we find that only $n = 1, 3$ seem to work. Let us prove that this is indeed the case.

Suppose $n \neq 1, 3$. Let $p$ be the minimal prime divisor of $n$. Then,

$$p \mid 2^n + 1 \implies p \mid 2^{2^n} - 1$$

Invoking Euler's theorem (really, Fermat's little theorem) we have that

$$p \mid 2^{p-1} - 1\,, \text{ hence } p \mid 2^{\gcd(p-1,2n)} - 1$$

Because $p$ is the *minimal* prime divisor of $n$, it follows that $\gcd(p-1, 2n) = 2$, so clearly $p \mid 2^2 - 1$, thus $p = 3$.

Now suppose that $3^k$ exactly divides $n$ (we will denote this by $3^k \parallel n$, a standard notation). So, $3 \parallel 2^2 - 1$ and by the famous exponent lifting so common in Olympiad problems we have that $3^{k+1} \parallel 2^{2n} - 1$.

If $n^2 \mid 2^n + 1$, then $3^{2k} \mid 2^{2n} - 1$, so $2k \leq k + 1 \implies k = 1$. So, 3 divides $n$ exactly.

Let $n = 3m$, $m \neq 1$, and let $q$ be the minimal prime divisor of $m$. Using the same argument as before, we have that

$$q \mid 2^{\gcd(q-1,6m)} - 1\,, \text{ so } \gcd(q-1, 6m) \in \{2, 6\}$$

Therefore, either $q \mid 2^2 - 1$ or $q \mid 2^6 - 1$. Recall that 3 divides $n$ exactly, so $q \neq 3$ and so $q$ must be 7. (Note that $2^6 - 1 = 7 \cdot 3^2$.)

But, $2^n + 1 = (2^3)^m + 1 \equiv 2 \pmod 7$, so 7 cannot possibly divide $2^n + 1$. We have arrived at a contradiction, thus proving the claim that $n = 1$ and $n = 3$ are the *only* solutions.

$$\boxed{n = 1, 3}$$

## Buffon's Needle Problem: *Solution*

The probability $P$ is the product of two others, namely $P_1$ or the probability that the center of the needle falls close enough to a line for a needle to cross it, and $P_2$ the probability that the needle actually crosses the line, given that the center is sufficiently close. A needle can cross a line if the center is within $\ell/2$ units of either side. So, we add the distance from both sides, $\frac{\ell}{2} + \frac{\ell}{2} = \ell$, then divide by the whole width, $w$: $P_1 = \frac{\ell}{w}$. Now we must calculate $P_2$.

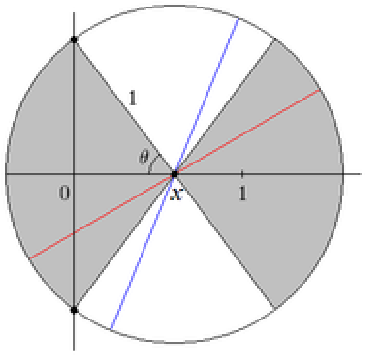
For simplicity, lets just say $\ell = 2$ for now. Let $x$ be the center of the needle and let $2\theta$ be the angle in which the needle, after having fallen, is contained in ($\theta$ is simply the angle between the vertical lines and the needle). If we set our coordinate system where the center of the needle lies on the $x$-axis and is in the direction of positive $x$, then the region of $2\theta$ is that in which the needle crosses the $y$-axis. There is a corresponding region to the right of the center. Nonetheless, because $\cos\theta = x$ we have that $\arccos(x) = \theta(x)$, and we need to find the proportion of the region where we have crossing to the total possible region.

If we just consider the left side, we have $\pi$ radian possible values and correspondingly have $\theta$ "good" values below the axis and another $\theta$ above, in other words the same $2\theta$ from before. To compare the proportions of areas we need to do none other than integrate. In particular,

$$P_2 = \int_0^1 \frac{2\theta(x)}{\pi}\,\mathrm{d}x = \frac{2}{\pi}\int_0^1 \cos^{-1}(x)\,\mathrm{d}x = \frac{2}{\pi}$$

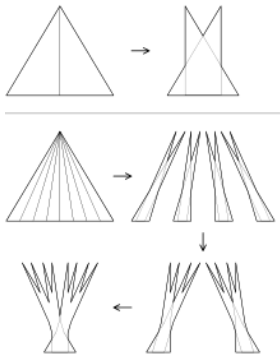
Therefore,

$$P = P_1 \cdot P_2 = \frac{\ell}{t}\frac{2}{\pi} = \boxed{\frac{2\ell}{t\pi}}$$



## Kakeya's Needle Problem: *Solution*

We use objects called "Perron trees" to do this in an intuitive way. The formal proof is very complex. We take a triangle of height 1, divide it into two parts down the middle, then translate both pieces over each other so they overlap and take our new figure to be that overlapping, which clearly is of a smaller area than the original triangle. Now, we could break our triangle into $n$ smaller triangles quite easily by simply drawing lines from the top vertex to some point on the base. We then do the same overlapping process for each sub-triangle. We overlap adjacent sub-triangles by shifting, then we overlap consecutive pairs of the new already overlapped sub-triangles, and we continue this overlapping process until we are left with one figure. **Because we reduce the area each time, we can make our area as small as we would like.**

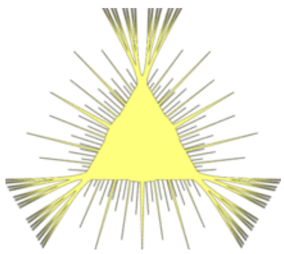


But, how can we possibly rotate a needle in an area such as this? Well, all would be well if we did not have to skip from one miniature region comprised of two overlapping triangles to another, but we can actually resolve that.

There is something called a Pal join, a trick of sorts, that will help us here. Given two parallel lines a distance $\epsilon$ apart, one can always rotate a unit segment by $180°$. Ultimately, the Perron tree we get at the end of the process described before will have tons of regions formed by overlapping miniature triangles, and these regions contain parallel lines, so we can use this trick to continuously rotate.

This trick's explanation can be intuitively thought of as taking two parallel lines, making them un-parallel by an arbitrarily small amount, $\epsilon$ where $\epsilon \to 0$, and then moving the needle arbitrarily far along the path of the two "parallel" lines until there is room to rotate.

Our final solution is a Kakeya needle set, which might look something like the below.



## Challenge Problem (Putnam 2002 A1): ***Solution***

We begin by differentiating the expression for the $n$th derivative, thus finding the $n+1$st derivative.

$$\left(\frac{P_n(x)}{(x^k-1)^{n+1}}\right)' = \frac{P_n'(x)(x^k-1)-(n+1)kx^{k-1}P_n(x)}{(x^k-1)^{n+2}}$$

$$P_{n+1}(x) = (x^k-1)P_n'(x)-(n+1)kx^{k-1}P_n(x)$$

If one lets $x=1$, then we have that

$$P_{n+1}(1) = -(n+1)kP_n(1)$$

But, $P_0(1)=1$, so we may use induction to find that

$$\boxed{P_n(1) = (-k)^n n!\ \forall n \geq 0}$$