2015-2016 Mathematics Competition Practice Session 4

Hagerstown Community College: STEM Club

November 6, 2015 12:00 pm - 1:00 pm STC-170

Beginner (Classic Mathematics Joke):

Three logicians walk into a bar. The barman says, "Does everybody want a drink?"

The first logician says, "I don't know."
The second logician says, "I don't know."

What does the third logician say?

Intermediate (The 3rd PGCC Math Tournament (2011), Round 1, Problem 3):

Two cultures of aliens live on the planet Trekia, the carpals and the tarsals. The carpals always lie, the tarsals always tell the truth.

A space traveler arrives on Trekia and meets a party of three aliens. She asks the aliens to which culture they belong. The first murmurs something that is too soft to hear. The second replies, "It said it was a carpal." The third says to the second, "you are a liar!"

From this info, figure out to what culture the third alien belongs.

Advanced (Singaporean Mathematics Competition):

Albert and Bernard just became friends with Cheryl, and they want to know when her birthday is. Cheryl gives them a list of 10 possible dates: May 15, 16, or 19; June 17 or 18; July 14 or 16; August 14, 15, or 17.

Cheryl then tells Albert and Bernard separately the month and the day of her birthday respectively.

The following conversation takes place:

Albert: "I don't know when Cheryl's birthday is, but I know that Bernard doesn't know too." Bernard: "At first I didn't know when Cheryl's birthday is, but I know now." Albert: "Then I also know when Cheryl's birthday is."

So when *is* Cheryl's birthday?

Discussion Questions:

Can a logical system simultaneously be complete and consistent?

Can we prove the consistency of math (in its ZFC formulation)?

How do we formally do logic?

Is mathematics based on logic?

SOLUTIONS:

Problem 1:

The key here is linguistic. The bartender asked if *everybody* wanted a drink, meaning all three persons would have to want one for the answer to be yes and only one would need to not want one for the answer to be no. So, the first logician must have wanted a drink, else he would have said no; however, because he cannot possibly know that the other two want a drink, he must say that he does not know. Similarly, the second logician does not know. Therefore, the last logician knows the former two want a drink. This means that **if the third logician wants a drink, the answer is yes, and if the third logician does not want a drink, the answer is no.**

■

Problem 2:

The first alien was to soft to hear, ergo the second alien could not have possibly heard the first say anything. Because the second alien said that the first *said* it was a carpal, and not that the first simply was a carpal, the second alien must be lying, for it did *not* hear the first say anything. Because the second alien lied, it must be a carpal. The third alien truthfully said the second alien lied, hence making it a **tarsal**.

■

Problem 3:

Albert cannot possibly know the answer, because there is no month with only one unique day in the list of dates, so that part of his statement is superfluous. But, because he knows that Bernard does not know either, we know that the month must be one that does not have a unique day associated with it, and the only unique days are the 18th and the 19th, which occur in June and May, respectively, hence the month cannot be May or June.

Bernard knows now that Albert has said he knows Bernard had not known previously. Therefore, Bernard must know that the day is one which is unique, considering only July and August because we have excluded the possibility of the date being in May or June. The only such unique days are the 15th, 16th, and 17th.

But, now Albert – who, let me remind you, knows the month – says he knows. Suppose the month were August, then there would be two possibly days, namely the 15th and the 17th that would be possibly (the 14th was excluded due to Bernard knowing the date). So, Albert would not possibly be able to know the exact date if the month were August, thus the month is July, and of the days previously considered to be possible, specifically the 15th, 16th, and 17th, the only one present in July is the 16th. Therefore, Cheryl's birthday is **July 16**.

■

NOTE: There are alternative answers to this that one might consider valid. [1]

Discussion Questions:

Our discussion questions were great and Joseph Heavner was very happy to present mathematical notions unbeknownst to other members as well as hear all the great thoughts and ideas others had, some of which are actually in the literature today.

The discussion was fairly natural, so splitting the discussion between questions is somewhat difficult, but below is roughly how it went.

Can a logical system simultaneously be complete and consistent?

No, this is known as Gödel's Incompleteness Theorem (one of two, really) [2], which states that any logical system strong enough to do arithmetic cannot be both complete and consistent.

Sometime around here I gave the continuum hypothesis [3] as an example of a theorem independent of our standard axiomatic system (see [14] for more such statements). This required introducing the ideas of cardinals [4] and Cantor's diagonal argument. [5] We also briefly discussed the hyperreals [6], because who doesn't line infinity? This then led to some discussion of the Banach-Tarski paradox [7] and whether nature is continuous.

I also briefly mentioned an example of a system known to be incomplete: classical Euclidean geometry. [8]

Can we prove the consistency of math (in its ZFC formulation)?

The conclusion was essentially "no". Joseph did not go into depth here, but one explore this further if one so desires. [9]

How do we formally do logic?

Symbols (quantifiers, *etc.*) and their formal definitions in terms of truth tables were presented. The idea of the contrapositive was presented and we proved that the contrapositive of any statement is logically equivalent to the statement itself. We also discussed the differences between contrapositive, converse, inverse, negation, and corollary. [10]

Is mathematics based on logic?

We discussed different possible bases of mathematics such as logic (canonical logic and intuitionist logic), set theory, category theory, and even some mention of homotopy type theory (HTT). [11]

At the end someone mentioned geometry and infinity, so I also presented the ideas of having a point or a line at infinity. [12] [13]

References

[1] https://www.youtube.com/watch?v=w3Nzkae_TRU

[2] http://mathworld.wolfram.com/GoedelsIncompletenessTheorem.html

[3] http://mathworld.wolfram.com/ContinuumHypothesis.html

[4] http://mathworld.wolfram.com/CardinalNumber.html

[5] http://mathworld.wolfram.com/CantorDiagonalMethod.html

[6] http://www.math.harvard.edu/~waffle/hyperreals.pdf

[7] http://mathworld.wolfram.com/Banach-TarskiParadox.html

[8] http://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/Greenberg2011.pdf

[9] https://en.wikipedia.org/wiki/Consistency

[10] http://www.jhtm.nl/tudelft/tw3520/Introduction_to_Mathematical_Logic.pdf

[11] https://en.wikipedia.org/wiki/Foundations_of_mathematics

[12] http://mathworld.wolfram.com/PointatInfinity.html

[13] http://mathworld.wolfram.com/LineatInfinity.html

[14] https://en.wikipedia.org/wiki/List_of_statements_undecidable_in_ZFC