2015-2016 Mathematics Competition Practice
Session 5

Hagerstown Community College: STEM Club

November 13, 2015   12:00 pm - 1:00 pm   STC-170

Warm-Up:

Suppose we have two circles C_1, C_2 each with radius $r = 5$ and a sphere S with radius $R = 12$ such that the circles lie on the surface of the sphere. At how many points is it possible for C_1, C_2 to intersect?

Perhaps consider this in more generality, for two circles C_1, C_2 and a sphere S.

Beginner (2007 AMC 12A No. 8):

A star-polygon is drawn on a clock face by drawing a chord from each number to the fifth number counted clockwise from that number. That is, chords are drawn from 12 to 5, from 5 to 10, from 10 to 3, and so on, ending back at 12. What is the degree measure of the angle at each vertex in the star polygon?

Intermediate (2013 AMC 10A No. 22):

Six spheres of radius 1 are positive so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of the eighth sphere?

NOTE: The two intersect at a point of tangency where the two spheres share a line of tangency. To be externally tangent is to be such that at every other point the two spheres lie on opposite sides of the line of tangency. Internally tangent means the exact opposite.

Advanced (2007 AIME II No. 15):

Four circles $\omega, \omega_A, \omega_B, \omega_C$ with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to the sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to $\omega_A, \omega_B, \omega_C$. If the sides of $\triangle ABC$ are $13, 14, 15$, the radius of ω can be represented in the form m/n for some relatively prime (*i.e.* sharing no factors) positive integers m, n. Find $m + n$.

Discussion Questions:

Is it possible to trisect an angle?

Is it possible to construct (in the Greek sense) a square with the same area as a circle?

Can a sphere be turned inside out using only smooth transformations, without cutting, tearing, or creasing?

SOLUTIONS

Warm-Up:

We have that the circumference of a great circle on the sphere is $2\pi R = 2\pi(10) = 20\pi$, and $4r = 4(5)$ is the combined diameter of the circles. Because the diameters cover less distance than the circumference of the sphere's great circle, we can place the circles on the sphere such that they do not intersect.Because of this, we can move the circles closer to have a point of tangency (one intersection), or even closer to intersect twice, or even have the circles placed on the same spot (infinitely many intersections). Hence the answer is

$$\boxed{\text{Number of Intersections} = 0, 1, 2, \infty}$$

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Beginner (2007 AMC 12A No. 8):

Consider just one vertex. If one draws a picture, one can easily see that the angle subtends one sixth of the circle, or 60°. Therefore, the angle itself is

$$\boxed{30^\circ}$$

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Intermediate (2013 AMC 10A No. 22):

Drawing a diagram for this can prove to be difficult, but one can find that the seventh circle surrounds all others and that the eighth is the one satisfying the tangency requirements provided. We construct an isosceles triangle between the center of the 8th sphere and the two opposite ends of the hexagon. We proceed by constructing a second triangle between the point of tangency of the 7th sphere and the 8th sphere, and the points of tangency between the 7th sphere and two of the initially identified spheres on the opposite sides of the hexagon. If we let r be the radius of the eighth sphere and h be the height of the first triangle, then we can use the Pythagorean theorem

$$(1 + r)^2 = 2^2 + h^2$$
$$(3\sqrt{2})^2 = 3^2 + (h + r)^2$$

Thus,

$$r = \boxed{\frac{3}{2}}$$

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Advanced (2007 AIME II No. 15):

This problem can be solved with some amount of ingenuity. By Heron's formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$ is the semiperimeter and a, b, c are side lengths, we have that $s = 21$ and so $A = \sqrt{(21(8)(7)(6)} = 84$. We can then find the inradius $A/s = 84/21 = 4$.

Now, the incenter I is not the same as the center of ω. If we take one of the small circle's radius to be r and say the three little circles tangent to the sides of $\triangle ABC$ have center X, Y, Z and we finally let P be the center of the circle tangent to the other three, then there exists a homothety centered at the incenter such that $X \mapsto A$, $Y \mapsto B$, and $Z \mapsto C$ with a dilation factor of $\frac{4-r}{4}$. This also maps P to the circumcenter of $\triangle ABC$, which means that, if we let R be the circumradius of $\triangle ABC$, $PX/R = 2r/R = (4-r)/4$. Hence, $R = a/(2\sin A) = 65/8$.

We finish with the following to solve for r:

$$\frac{2r}{\frac{65}{8}} = \frac{4-r}{4} \implies r = \frac{260}{129} \implies m+n = 260+129 = \boxed{389}$$

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Discussion Questions:

When writing this up, it was assumed that people would remember their geometry courses and Euclidean constructions, but that was not the case. As such, the first two questions were hardly discussed, but it was noted that in both cases the answer is "no", as can be shown using abstract algebra. [1] [2]

The third question's solution is not particularly easy to understand or explain, but it turns out to be "yes", as was proved by Smale in 1958. [3] There is a very interesting video showing how to do this explicitly. [4]

References

[1] http://sites.mathdoc.fr/JMPA/PDF/JMPA_1837_1_2_A31_0.pdf

[2] http://fan2cube.fr/mathtador/pdf/lindermann-weierstrass.pdf

[3] https://www.jstor.org/stable/1993205?seq=1#page_scan_tab_contents

[4] https://www.youtube.com/watch?v=wO61D9x61NY