2015-2016 Mathematics Competition Practice
Session 6

Hagerstown Community College: STEM Club

November 20, 2015   12:00 pm - 1:00 pm   STC-170

Warm-Up (2006 AMC 10B No. 17):

Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?

Beginner (2010 AMC 12B No. 17):

The entries in a 3×3 array include all the digits from 1 to 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

Intermediate (2013 AIME II No. 9):

A 7×1 board is completely covered by $m \times 1$ tiles without overlap; each tile may cover any number of consecutive squares, and each tile lies completely on the board. Each tile is either red, blue, or green. Let N be the number of tilings of the 7×1 board in which all three colors are used at least once. For example, a 1×1 red tile followed by a 2×1 green tile, a 1×1 green tile, a 2×1 blue tile, and a 1×1 green tile is a valid tiling. Note that if the 2×1 blue tile is replaced by two 1×1 blue tiles, this results in a different tiling. Find the remained when N is divided by 1000.

Advanced (2007 AIME II No. 10):

Let S be a set with six elements. Let P be the set of all subsets of S. Subsets A and B of S, not necessarily distinct, are chosen independently and at random from P. The probability that B is contained in at least one of A or $S - A$ is $\frac{m}{n^r}$ where m, n, r are positive integers, n prime such that m and n are coprime. FInd $m + n + r$. (The set $S - A$ is the set of all elements in S which are not in A.)

Discussion Questions:

Consider a room of n random people. For what value of n do we have that there is some pair of two people in the room that have the same birthday? *(Birthday Problem)*

Suppose you are given the choice of three doors to choose: Behind one door is a car and behind the others are goats. You pick a door, say door number 1, and the host, who knows what is behind each of the doors, opens another door, say door 2, which has a goat. He then asks "Do you want to change your choice to door 3?" Is it advantageous to change your choice from door 1 to door 3? *(Monty-Hall Problem)*

What is the probability that if one chooses a number randomly from the real line, that number will be the number 0? *(Infinite probabilities)*

In sets of numerical data, at what frequency will the leading digit of a number be 1? *(Benford's Law)*

What is Pascal's wager? *(Pascal's wager)*

Suppose we have three rods and a number of different sized disks which can slide onto any rod. Our initial state has the disks on a single rod from largest to smallest. If we can only move one disk at a time, each move consists of taking the upper disk from one stack and placing it on top of another stack, and no disk can be placed on top of a smaller disk, then what is the minimum number of moves required to solve the puzzle? *(Tower of Hanoi puzzle)*

Suppose we have a graph G (*Definition: a representation of a set of objects where pairs of objects are connected).* If G is a path, *i.e.* it connects distinct vertices, and G only visits each vertex once, it is called a *Hamiltonian path.* What is a neccessiary and sufficient condition for a graph with $n \geq 3$ vertices to be Hamiltonian? *(Dirac's Theorem)*

(A related path is the *Eulerian path*, which visits every edge once.)

SOLUTIONS

Warm-Up (2006 AMC 10B No. 17):

Alice's choice is irrelevant. Without loss of generality, suppose Alice selected a red ball and placed it into Bob's bag. Bob now has six balls. In order for each bag to have the same contents, Bob must put a red ball into Alice's bag. Bob's bag consists of 2 red balls, 1 blue ball, 1 green ball, 1 orange ball, and 1 violet ball. Therefore,

$$\mathbb{P}\,(\text{Red}) = \frac{2}{6} = \boxed{\frac{1}{3}}$$

Beginner (2010 AMC 12B No. 17):

There exists a theorem, the so-called hook length theorem from combinatorics, which provides us with an immediate answer. However, this is obscure, and so we will take a different route.

One can easily convince oneself that the numbers 1 through 4 have but three possible configurations, up to reorganization. One is a block, which we will denote B. There are two possible blocks, which are relocated below.

1	2	
3	4	

1	3	
2	4	

Another possible configuration is an L-shape, which we will naturally denote L. One can check that there exist three permutations of L.

1	2	3
4		

1	2	4
3		

1	3	4
2		

The final possible configuration of the first four numbers is a J-shape, which we will denote J. Again, there exist three permutations.

1	4	
2		
3		

1	3	
2		
4		

1	2	
3		
4		

One can clearly see that if we consider $5-9$ we have that $6-9$ form the same shapes, namely O, L, J, up to rotation and translation, that is.

So, all we need is to pair each configuration of $1-4$ with each of $6-9$. (The 5 spot falls out of the problem.)

$1-4$ shape	$6-9$ shape	number of pairings
O	O	$2\times 2=4$
O	J	$2\times 3=6$
O	L	$2\times 3=6$
J	J	$3\times 3=9$
J	L	$3\times 3=9$
L	O	$3\times 2=6$
L	L	$3\times 3=9$
L	J	$3\times 3=9$
L	O	$3\times 2=6$

Obviously, the O/O, J/J, and L/L situations are not possible, so we can ignore those.

$$\#\text{ of Arrays}: 6+6+9+6+9+6=\boxed{42}$$

Intermediate (2013 AIME II No. 9):

Let us consider the number of ways we can tile the board given some nuber of pieces. If we have 3 pieces, then we can cover with a 5-piece and 2 1-pieces, a 4-piece and a 2-piece, (where a k-piece is a piece of length k) *etc.*. It is easy to see that the total number of ways to do this is $\binom{6}{2}=15$.Note that there is no reason to consider tiling the board with only 1 or 2 pieces, because we must use each of the three colors.

Similarly, we have that for four pieces the number of ways is $\binom{6}{3}=20$; for five pieces we have $\binom{6}{4}=15$ ways; for six pieces we have $\binom{6}{5}=6$ ways; for seven pieces we have $\binom{6}{6}=1$ ways. Clearly, we cannot use more than 7 pieces.

We now make use of the *inclusion-exclusion principle*. As such, we will briefly review the theorem.

The statement of this can be written abstractly as

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\varnothing\neq J\subseteq\{1,2,\ldots,n\}} (-1)^{|J|-1}\left|\bigcap_{j\in J} A_j\right|$$

But, this is not very helpful. Instead, let us explain this intuitively. If we want to find the size of n sets, then we take the cardinality of the union of all the sets (this includes their overlaps), then we exclude the cardinalities of the pairwise intersections, then we include the cardinalities of the triple-wise intersections, and proceed onward, including all odd n and excluding all even n. One can easily see this for $n=2,3$.

Therefore, for three pieces we have: $3^3-3\cdot 2^3+3=6$ colorings, and for four pieces we have $3^4-3\cdot 2^4+3=36$, and similarly we have 150 for five pieces, 540 for six pieces, and 1806 for seven pieces.

Now, we multiply the number of colorings by the number of ways we can cover the board with n pieces.

$$(6\cdot 15)+(36\cdot 20)+(150\cdot 15)+(540\cdot 6)+(1806\cdot 1)=8106$$

Now, read the question closely once more. We need to take that number, N, and find its remainder after dividing by 1000. So, our answer is just

$$\boxed{106}$$

Advanced (2007 AIME II No. 10):

Suppose B has 6 elements, hence the probability is $1/2^6 = 1/64$, then A must have either 0 or 6 elements, giving probability $2/64$. Suppose now that B has 5 elements, then the probability is $\binom{6}{5}/64 = 15/64$. Here A must have 0,6, or 1,5 elements. The total probability is therefore additive and given by $2/64 + 2/64 = 4/64$. In general, we have

$$\mathbb{P}\left(\text{choosing } B \text{ with } n \text{ elements}\right) = \frac{\binom{6}{n}}{64}$$

The sum of the elements in the kth row of Pascal's triangle is 2^k, so the probability of obtaining A or $S - A$ that contains B is

$$\mathbb{P}\left(\text{obtaining } A \text{ or } S\text{-}A \text{ containing } B\right) = \frac{2^{7-n}}{64}$$

We must not forget about $B = \varnothing$, either. (This is the final term in the sum below.) Given all this, we have the following:

$$\begin{aligned}
\frac{m}{n^r} &= \left(\sum_{i=1}^{6} \frac{\binom{6}{i}}{64} \cdot \frac{2^{7-i}}{64}\right) + \frac{1}{64} \cdot \frac{64}{64} \\
&= \frac{(1)(64) + (6)(64) + (15)(32) + (20)(16) + (15)(8) + (6)(4) + (1)(2)}{(64)(64)} \\
&= \frac{1394}{2^{12}} \\
&= \frac{697}{2^{11}}
\end{aligned}$$

Thus, our solution is

$$m + n + r = 627 + 2 + 11 = \boxed{710}$$

Discussion Questions:

Because we did not actually meet this time due to a last minute cancellation, I have simply provided the names of the theorems, *etc.* relevant to the discussion problems (names are provided at the end of each question). If one seems interesting to you, perhaps google the name get more information, after you explore it yourself, of course.